

Cosets

G a gp, $H \leq G$ a subgp.

$a \sim b$ if $\exists h \in H$ so that $b = ah$.

This is an equivalence reln

Defⁿ $aH := \{ah : h \in H\}$
 $a \in G$

These sets aH are called the left cosets of H in G .

Similarly, $Ha = \{ha : h \in H\}$. These sets Ha are called the right cosets of H in G .

Cor^y The left cosets of H in G partition G .

Pf. The cosets aH are the equivalence classes for an equivalence reln $a \sim b$ if $b = ah$.

Defⁿ The index of a subgp H in a gp G is the # of left cosets aH in G . This number is denoted $[G : H]$.

Fact: $H \leq G$ a subgp. Then the # of (left cosets of H in G) is the same as the number of right cosets.

Pf. Consider the map $\text{inv}: G \rightarrow G$.

Takes

$$\boxed{\text{inv}} \left(\underbrace{aH}_{\text{in } G} \right) \longrightarrow \underbrace{Ha^{-1}}_{\text{in } G}$$

(using that H is closed under inverse)

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unter inverse)

This bijection $g \mapsto hg$ induces a bijection between left cosets and right cosets.

It is a bijection because this map on cosets is invertible. $\square$

Propⁿ All left cosets all have the same order, the order of H .

Pf: $L_a: I \cdot H \longrightarrow aH$ $a \in G$ arbitrary
 $h \longmapsto ah$

is a bijection $\implies |H| = |aH|$. $\blacksquare$

$$Q = \bigcup_a aH \quad \text{cosets}$$

$$|a| = \sum_a |a_H| \quad \approx \quad [a:H] |H| = |a|$$

Thm (Lagrange's Thm) Suppose $|G|$ is finite. Then if $H \subseteq G$ is a subgp, $|H|$ divides $|G|$.

Defⁿ G a gp, $g \in G$ an elt. The order of g , $\text{ord}(g)$ is the smallest positive integer n so that $g^n = 1$.

Fact: Denote by $\langle g \rangle$ the cyclic subgp of G gen'd by g .
Then $|\langle g \rangle| = \text{ord}(g)$.

$$\langle a \rangle = \int_1^{\infty} a \sigma^2 \cdot a^{m-1} da$$

$$\langle g \rangle = \{1, g, g^2, \dots, g^{m-1}\}$$

Cor: If $g \in G$, then $\text{ord}(g)$ divides $|G|$.

Ex If $G = S_3$, $|S_3| = 3! = 6$.

eg: $x = (123)$, $x^3 = 1$ $\text{ord}(x) = 3$ and $3|6$.

$y = (12)$, $y^2 = 1$ $\text{ord}(y) = 2$ and $2|6$.

Cor Suppose $|G| = p$ is prime. Then, G is cyclic of order p . In particular, G is abelian.

Pf: Take $g \in G$ with $g \neq 1$.

Consider $\langle g \rangle \leq G$.

Have: $|\langle g \rangle|$ divides $|G| = p$.

$\Rightarrow |\langle g \rangle| = p$ and thus $\langle g \rangle = G$. $\square$

Lemma $\uparrow$ $\varphi: G \rightarrow G'$ is a gp homom, $K := \ker(\varphi)$.

Suppose

Then $\varphi(a) = \varphi(b) \Leftrightarrow b = ak \Leftrightarrow bK = aK$.

for some
 $k \in K$

Pf $\varphi(a) = \varphi(b) \Leftrightarrow \varphi(a^{-1}b) = 1 \Leftrightarrow a^{-1}b \in K$

$\Leftrightarrow b \in aK \Leftrightarrow bK = aK$. $\square$

Cor: $\varphi: G \rightarrow G'$ a sp homom. Then

$$|\text{Im}(\varphi)| = [G : \ker(\varphi)]$$

Pf: The size of $\text{Im}(\varphi)$ is exactly the # of nonempty fibers of $\varphi: G \rightarrow G'$.

But, this is the number of cosets of $\ker(\varphi)$ in G . $\blacksquare$

Cor: Suppose $\varphi: G \rightarrow G'$ is a sp homom. Then

$$\begin{aligned} |G| &= [G : \ker(\varphi)] |\ker(\varphi)| \\ &= |\text{Im}(\varphi)| \cdot |\ker(\varphi)|. \end{aligned}$$

$\Rightarrow |\ker(\varphi)|$ divides $|G|$

• $|\text{Im}(\varphi)|$ divides $|G|$ and $|G'|$.

Example Suppose $\varphi: G \rightarrow G'$ is a homom. of fte gps and $\gcd(|G|, |G'|) = 1$

Then: $\varphi: G \rightarrow G'$ is the trivial map, i.e. $\varphi(g) = 1$ for all $g \in G$.

Pf: $|\text{Im}(\varphi)|$ divides $|G|$ and $|G'|$

$$\Rightarrow |\text{Im}(\varphi)| = 1 \Rightarrow \varphi(g) = 1 \quad \forall g.$$

Propⁿ Suppose $G \supseteq H \supseteq K$ subgps of G .

$$\text{Then } [G : K] = [G : H][H : K]$$

Pf: Suppose $[G : H] = m$ and $[H : K] = n$.

$G = a_1 H \cup a_2 H \cup \dots \cup a_m H$ is a partition of G

$G = g_1 H \cup g_2 H \cup \dots \cup g_m H$ is a partition of G .

$H = h_1 K \cup h_2 K \cup \dots \cup h_n K$ is a partition of H .

$\Rightarrow \underline{g_j H = g_j h_1 K \cup g_j h_2 K \cup \dots \cup g_j h_n K}$ is a partition of $g_j H$ into K cosets.

b/c left multiplication by g_j is an invertible map.

$$H \rightarrow g_j H$$

$\Rightarrow G = \bigcup_{\substack{1 \leq j \leq m \\ 1 \leq l \leq n}} g_j h_l K$ is a partition of G into mn cosets of K . $\square$

Example (of left vs right cosets)

$$G = S_3 \quad x = (123), y = (12)$$

$$H = \{1, y\}$$

Left cosets of H in G

$$H = \{1, y\}, \quad xH = \{x, xy\}, \quad x^2 H = \{x^2, x^2 y\} \quad |$$

$$G = \{1, x, x^2, y, xy, x^2 y\}$$

Right cosets of H in G

$$H = \{1, y\}, \quad Hx = \{x, yx\} = \{x, x^2 y\}, \quad Hx^2 = \{x^2, yx^2\} = \{x^2, xy\} \quad ||$$

The right cosets give a diff^t partition than the left cosets